Exercise 30

Solve the boundary-value problem, if possible.

$$4y'' - 4y' + y = 0, \quad y(0) = 4, \quad y(2) = 0$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad y' = re^{rx} \quad \rightarrow \quad y'' = r^2 e^{rx}$$

Plug these formulas into the ODE.

$$4(r^2 e^{rx}) - 4(re^{rx}) + e^{rx} = 0$$

Divide both sides by e^{rx}.

$$4r^2 - 4r + 1 = 0$$

Solve for r.

$$(2r - 1)^2 = 0$$

$$r = \left\{ \frac{1}{2} \right\}$$

Two solutions to the ODE are $e^{x/2}$ and $xe^{x/2}$. By the principle of superposition, then,

$$y(x) = C_1 e^{x/2} + C_2 x e^{x/2}.$$

Apply the boundary conditions to determine C_1 and C_2.

$$y(0) = C_1 = 4$$

$$y(2) = C_1 e + 2C_2 e = 0$$

Solving this system of equations yields $C_1 = 4$ and $C_2 = -2$. Therefore, the solution to the boundary value problem is

$$\begin{aligned} y(x) &= 4e^{x/2} - 2xe^{x/2} \\ &= 2e^{x/2}(2 - x). \end{aligned}$$

Below is a graph of $y(x)$ versus x.